COMMENT ON THE ARTICLE BY V. V. IVANOV "TEMPERATURE DISTRIBUTION IN A TOROIDAL COIL HEATED BY ELECTRIC CURRENT" AND ON THE CORRECT SOLUTION TO THE PROBLEM

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The problem concerning the temperature field of a coil heated by electric current is reduced in [1] to solving the Poisson equation $\Delta T + (q_v/\lambda) = 0$ with the boundary condition $\lambda(\partial T/\partial n) + hT = 0$* on the torus surfaces. Here Δ is the Laplace operator, T is the temperature, q_v is the thermal flux density generated by the electric current per unit time, λ is the thermal conductivity of the coil material, and h is the heat transfer coefficient. The solution is constructed in a toroidal system of coordinates α, β which are related to cylindrical coordinates z, ρ as follows [2, 3]:

$$\rho = \frac{c\,\mathrm{sh}\,\alpha}{\mathrm{ch}\,\alpha - \cos\beta}, \quad z = \frac{c \sin\beta}{\mathrm{ch}\,\alpha - \cos\beta} \quad (0 \leqslant \alpha < \infty, -\pi < \beta \leqslant \pi),$$

where c is a scale factor.

By the substitution

$$T(\alpha, \beta) = U(\alpha, \beta) - \frac{q_v c^2\,\mathrm{sh}^2\,\alpha}{4\lambda(\mathrm{ch}\,\alpha - \cos\beta)^2} \tag{1}$$

the problem is reduced to finding function U which satisfies the Laplace equation $\Delta U = 0$ and the boundary conditions at the torus surface $\alpha = \alpha_0$

$$(\mathrm{ch}\,\alpha_0 - \cos\beta)\left[\frac{\partial U}{\partial \alpha} - \frac{q_v}{2\lambda}c^2\frac{\mathrm{sh}\,\alpha_0(1-\mathrm{ch}\,\alpha_0\cos\beta)}{(\mathrm{ch}\,\alpha_0-\cos\beta)^2}\right] = \mathrm{Bi}\left[U - \frac{q_v}{4\lambda}c^2\frac{\mathrm{sh}^2\alpha_0}{(\mathrm{ch}\,\alpha_0-\cos\beta)^2}\right], \quad \mathrm{Bi} = \frac{hc}{\lambda}. \tag{2}$$

After separation of variables, function U can be written as

$$U = \sqrt{2\,\mathrm{ch}\,\alpha - 2\cos\beta}\sum_{n=0}^{\infty} A_n Q_{n-1/2}(\mathrm{ch}\,\alpha)\cos n\beta \quad (0 < \beta < \pi), \tag{3}$$

where $Q_{n-1/2}(t)$ are spherical functions and A_n are unknown numbers yet to be determined. Inserting this expression for U into (2), in order to determine A_n, yields

$$\sum_{n=0}^{\infty} A_n Q_{n-1/2}\frac{\cos n\beta}{\sqrt{2\,\mathrm{ch}\,\alpha_0 - 2\cos\beta}} + \frac{2\,\mathrm{ch}\,\alpha_0 - 2\cos\beta}{\sqrt{2\,\mathrm{ch}\,\alpha_0 - 2\cos\beta}}\sum_{n=0}^{\infty} A_n Q'_{n-1/2}\cos n\beta$$

$$-\frac{q_v}{2\lambda}c^2\frac{1-\mathrm{ch}\,\alpha_0\cos\beta}{(\mathrm{ch}\,\alpha_0-\cos\beta)^{5/2}}\;\frac{1}{\sqrt{\mathrm{ch}\,\alpha_0-\cos\beta}} = \mathrm{Bi}\frac{2}{\mathrm{sh}\,\alpha_0}\sum_{n=0}^{\infty} Q_{n-1/2}A_n$$

*In [1] this condition is stated in the form $\lambda\,\mathrm{grad}\,T = hT$. Such a notation is inaccurate on two counts: it does not indicate that the projection of vector grad T on the outward normal to the torus is under consideration here and, secondly, the minus sign before h is omitted. The second error is compensated later by omission of the minus sign before $\partial T/\partial\alpha$ in the boundary conditions.

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$$\times \frac{\cos n\beta}{\sqrt{2\,\mathrm{ch}\,\alpha_0 - 2\cos\beta}} - \mathrm{Bi}\frac{q_v}{4\lambda}c^2 \frac{\mathrm{sh}\,\alpha_0}{(\mathrm{ch}\,\alpha_0 - \cos\beta)^{5/2}} \frac{1}{\sqrt{\mathrm{ch}\,\alpha_0 - \cos\beta}}, \tag{4}$$

where, for simplicity, $Q_{n-1/2} \equiv Q_{n-1/2}(\cosh\alpha_0)$. Expression (4) has been derived in [1]. It is correct. Inserting

$$\frac{1}{\sqrt{2\,\mathrm{ch}\,\alpha_0 - 2\cos\beta}} = \sum_{n=0}^{\infty} \frac{\varepsilon_n}{\pi} Q_{n-\frac{1}{2}}(\mathrm{ch}\,\alpha_0)\cos n\beta, \quad \varepsilon_n = \begin{cases} 1 & \text{for } n = 0, \\ 2 & \text{for } n = 1, 2, 3\ldots \end{cases}$$

into those terms in (4) where there is no summation sign and then integrating from 0 to π, Ivanov without justification equates terms with the same subscript n and thus arrives at the wrong conclusion. The correct procedure is to determine A_n in the usual manner, as follows.

We multiply (4) by $\sqrt{2\cosh\alpha_0 - 2\cos\beta}$, introduce new unknowns

$$X_n = A_n\, Q'_{n-1/2}(\mathrm{ch}\,\alpha_0), \tag{5}$$

and obtain

$$\sum_{n=0}^{\infty} X_n d_n \cos n\beta - 2\cos\beta \sum_{n=0}^{\infty} X_n \cos n\beta - \frac{2q_v}{\lambda}c^2 \left[\frac{2 - \mathrm{Bi}\,\mathrm{sh}\,\alpha_0}{(2\,\mathrm{ch}\,\alpha_0 - 2\cos\beta)^{5/2}} - \frac{2\,\mathrm{ch}\,\alpha_0 \cos\beta}{(2\,\mathrm{ch}\,\alpha_0 - 2\cos\beta)^{5/2}}\right] = 0 \quad (0 < \beta < \pi), \tag{6}$$

where

$$d_n = \frac{Q_{n-1/2}(\mathrm{ch}\,\alpha_0)}{Q'_{n-1/2}(\mathrm{ch}\,\alpha_0)}\left(1 - \mathrm{Bi}\frac{2}{\mathrm{sh}\,\alpha_0}\right) + 2\,\mathrm{ch}\,\alpha_0.$$

Function β on the left-hand side of Eq. (6) will be equal to zero, if all coefficients of its series expansion in $\cos m\beta$ are equal to zero. These coefficients are found by multiplying (6) by $(\varepsilon_m/\pi)\cos m\beta$ ($m = 0, 1, 2, \ldots$) and subsequently integrating from 0 to π.

Performing the necessary operations and considering that

$$\int_0^{\pi} \frac{\cos m\beta\, d\beta}{(2\,\mathrm{ch}\,\alpha_0 - 2\cos\beta)^{5/2}} = \frac{1}{3} Q''_{m-1/2}(\mathrm{ch}\,\alpha_0),$$

we obtain an infinite system of linear algebraic equations in the unknown variables X_n:

$$X_m = \frac{\frac{2}{\varepsilon_m} X_{m+1} + (\varepsilon_m - 1) X_{m-1}}{d_m} + b_m \quad (m = 0, 1, 2\ldots), \tag{7}$$

where

$$b_m = \frac{2q_v c^2 \varepsilon_m}{3\pi\lambda d_m}\left[(2 - \mathrm{Bi}\,\mathrm{sh}\,\alpha_0)\, Q''_{m-1/2}(\mathrm{ch}\,\alpha_0) - \mathrm{ch}\,\alpha_0 \left(\frac{2}{\varepsilon_m} Q''_{m+1/2}(\mathrm{ch}\,\alpha_0) + Q''_{m-3/2}(\mathrm{ch}\,\alpha_0)(\varepsilon_m - 1)\right)\right].$$

We note that the system has nonzero matrix elements on the principal diagonal and on the adjoining ones.

Considering that $Q_{m-1/2}(\cosh\alpha_0)/Q'_{m-1/2}(\cosh\alpha_0)$ and $Q''_{m-1/2}(\cosh\alpha_0)$ tend toward zero as m increases [2], it is easy to prove that $b_m \to 0$ when $m \to \infty$ and that, beginning at some $m = m_0$, the sum of the coefficients of the unknowns $|2/d_m|$ is smaller than $\delta < 1$. This means [4] that the infinite system (7) is completely quasiregular and can be solved by the method of successive reductions. The value m_0 depends on h, c, λ, α_0 and can easily be determined in every specific case. The values of X_m found from (7) with (5), (3), and (1) taken into account yield, unlike [1], the correct solution to the problem.

LITERATURE CITED

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